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Pseudo-Boson Coherent and Fock States

D.A. Trifonov *
Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
72 Tzarigradsko chausse Blv, 1784 Sofia, Bulgaria

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Abstract

Coherent states (CS) for non-Hermitian systems are introduced as eigenstates of pseudo-Hermitian boson annihilation operators. The set of these CS includes two subsets which form bi-normalized and bi-overcomplete system of states. The subsets consist of eigenstates of two complementary lowering pseudo-Hermitian boson operators.

Explicit constructions are provided on the example of one-parameter family of pseudo-boson ladder operators. The wave functions of the eigenstates of the two complementary number operators, which form a bi-orthonormal system of Fock states, are found to be proportional to new polynomials, that are bi-orthogonal and can be regarded as a generalization of standard Hermite polynomials.

1 Introduction

In the last decade a growing interest is shown in the non-Hermitian PT-symmetric (or pseudo-Hermitian) quantum mechanics. For a review with an enlarged list of references see the recent papers [1, 2]. This trend of interest was triggered by the papers of Bender and coauthors [3], where the Bessis conjecture about the reality and positivity of the spectrum of Hamiltonian $H = p^2 + x^2 + ix^3$ was proven ('using extensive numerical and asymptotic studies') and argued that the reality of the spectrum is due to the unbroken PT-symmetry. The Bessis–Zinn Justin conjecture about the reality of the spectrum of the PT-symmetric Hamiltonian $p^2 - (ix)^{2\nu}$ for $\nu \geq 1$ has been proven in Ref. [4]. A criterion for the reality of the spectrum of non-Hermitian PT-symmetric Hamiltonians is provided in Ref. [5]. Mustafazadeh [6] has soon noted that all the PT-symmetric non-Hermitian Hamiltonians studied in the literature belong to the class of pseudo-Hermitian

*e-mail: dtrif@inrne.bas.bg

Hamiltonians. A Hamiltonian H was said to be pseudo-Hermitian if it obeys the relation [6]

$$H^{\#} := \eta^{-1} H^{\dagger} \eta = H, \tag{1}$$

where η is an invertible Hermitian operator. $H^{\#}$ was called *η-pseudo-Hermitian conjugate* to H, shortly *η-pseudo-adjoint* to H. The PT-symmetric Hamiltonian $H = p^2 - (ix)^{2\nu}$, examined in Refs. [3, 4], obeys (1) with η equal to the parity operator P. The spectrum of a diagonalizable pseudo-Hermitian H is either real or comes in complex conjugate pairs. A diagonalizable (non-Herimitian) Hamiltonian H has a real spectrum iff it is pseudo-Hermitian with respect to positive definite η [7]. In terms of P and T operations the reality of the spectrum of H occurs if the PT symmetry is exact (not spontaneously broken) (see e.g. Refs. [1, 2] and references therein). In fact many of the later developments in the field are anticipated in the paper by Scholtz et al [8] (see comments in Ref. [1]).

In the present paper we address the problem of construction of pseudo-Hermitian boson (shortly pseudo-boson) creation and annihilation operators and related Fock states and coherent states (CS). For pseudo-fermion system ladder operator CS have been introduced by Cherbal et al [11] on the example of two-level atom interacting with a monochromatic em field in the presence of level decays. For non-Hermitian PT-symmetric system CS of Gazeau-Klauder type have been constructed, on the example of Scarf potential, by Roy et al [9]. Annihilation and creation operators in non-Hermitian (supersymmetric) quantum mechanics were considered by Znojil [13]. For the bosonic PT symmetric singular oscillator (which depicts a double series of real energy eigenvalues) [14] ladder operators and eigenstates of the annihilation operators have been built up by Bagchi and Quesne [15]. Our main aim here is the construction of overcomplete families (in fact bi-overcomplete) of ladder operator CS for pseudo-bosons. The problem of pseudo-boson ladder operators is considered in section 2. In the third section we consider the construction of eigenstates of pseudo-boson number operators. The pseudo-boson CS are introduced and discussed in section 4. Explicit example of pseudo-boson ladder operators and related eigenstates is provided and briefly commented in section 5. Outlook over the main results is given in the Conclusion.

2 Pseudo-boson ladder operators

With the aim to construct pseudo-Hermitian boson (shortly *pseudo-boson*) coherent states (CS) we address the problem of ladder operators that are *pseudo-adjoint* to each other. In analogy with the pseudo-Hermitian fermion (phermion) annihilation and creation operators [10] the η-pseudo-boson ladder operators $b,\ b^{\#} := \eta^{-1} b^{\dagger} \eta$ can be defined by means of the commutation relation

$$[b, b^{\#}] \equiv b b^{\#} - b^{\#} b = 1. \tag{2}$$

If $\eta = 1$ the standard boson operators a, $a^{\dagger}$ are recovered.

From (2) it follows that the pseudo-Hermitian (pseudo-selfadjoint) operator $b^{\#}b \equiv N$ commutes with b and $b^{\#}$ according to

$$[b, N] = b, \quad [b^{\#}, N] = -b^{\#}, \tag{3}$$

and could be regarded as *pseudo-boson number operator*. For a pair of non-Hermitian operators b, $\tilde{b}$ with commutator $[b, \tilde{b}] = 1$, the existence of η such that $\tilde{b} = \eta^{-1} b^{\dagger} \eta \equiv b^{\#}$, stems from the existence of b-vacuum. We have the following

Proposition 1. If the operators b and $\tilde{b}$ and a state $|0\rangle$ satisfy

$$[b, \tilde{b}] = 1, \quad b|0\rangle = 0, \tag{4}$$

then $\tilde{b}$ is η-adjoint to b with

$$\eta = \sum_{n=0} |\varphi_n\rangle\langle\varphi_n|, \tag{5}$$

where $|\varphi_n\rangle$ are eigenstates of $N' = b^{\dagger}\tilde{b}^{\dagger}$.

Proof. Note first that the non-Hermitian operator $N = \tilde{b}b$ is diagonalizable and with real and discrete spectrums. It's eigenstates can be constructed acting on the b-vacuum by the operators $\tilde{b}$ correspondingly (see the next section). The spectrum of a diagonalizable non-Hermitian operator H is real iff H is η-pseudo-Hermitian (theorem of Ref. [6]), and this η may be chosen as a sum of projectors onto the eigenstates of $H^{\dagger}$ [6]. In our cases $H = N$ and $H^{\dagger} = N^{\dagger} = b^{\dagger}\tilde{b}^{\dagger}$. This ends the proof of the Proposition 1.

For the sake of completeness however we provide in the next section the construction (and brief discussion) of the eigenstates of N, N'.

Remark. A similar Proposition can be formulated and proved for a pair of non-Hermitian nilpotent operators g, $\tilde{g}$, $g^2 = 0$ with anticommutator $\{g, \tilde{g}\} = 1$: just replace b, $\tilde{b}$, $[b, \tilde{b}]$ with g, $\tilde{g}$, $\{g, \tilde{g}\}$.

3 Pseudo-boson Fock states

The eigenstates of pseudo-boson number operators $N = \tilde{b}b$ can be constructed acting on the ground states $|0\rangle$ by the raising operator $\tilde{b}$:

$$|\psi_n\rangle = \frac{1}{\sqrt{n!}} \tilde{b}^{\,n} |0\rangle, \tag{6}$$

$$N|\psi_n\rangle = \tilde{b}b|\psi_n\rangle = n|\psi_n\rangle. \tag{7}$$

However in view of $\tilde{b} \neq b^{\dagger}$ these number states are not orthogonal. It turned out that a *complementary pair* of pseudo-boson ladder operators and number operator exist, such that the system of the two complementary sets of number states form the so-called *bi-orthogonal and bi-complete sets*. Indeed, if $\tilde{b}$ is

creating operator related to b, then, on the symmetry ground, we could look for new operators b' for which $b^\dagger$ is the creating operator,

$$[b', b^\dagger] = 1. \tag{8}$$

The pairs of "prime"-ladder operators b', $b'^\dagger$ is just $\tilde{b}^\dagger$, $b^\dagger$, and the "prime" number operator is

$$N' = b'^\dagger b' = b^\dagger \tilde{b}^\dagger. \tag{9}$$

The eigenstates of N' are constructed in a similar way acting with $b^\dagger$ on the b'-vacuums $|0\rangle'$:

$$|\varphi_n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle'. \tag{10}$$

The existence of the b'-vacuum $|0\rangle' = |\varphi_0\rangle$ follows from the properties of the pseudo-Hermitian operators H with real spectra [6, 7]: the spectrum of H and $H^\dagger$ coincide since they are related via a similarity transformation. In our case $H = \tilde{b}b$, $H^\dagger = b^\dagger\tilde{b}^\dagger \equiv b^\dagger b'$.

Using the commutation relations of the above described ladder operators one can easily check that if $\langle 0|0\rangle' = 1$ then the "prime" number-states $|\varphi_n\rangle$ are *bi-orthonormalized* to $|\psi_n\rangle$ (that is $\langle\psi_n|\varphi_m\rangle = \delta_{nm}$), and form together the *bi-complete* system of states $\{|\psi_n\rangle, |\varphi_n\rangle\}$:

$$\sum_n |\psi_n\rangle\,\langle\varphi_n| = 1 = \sum_n |\varphi_n\rangle\,\langle\psi_n|. \tag{11}$$

The set $\{|\psi_n\rangle, |\varphi_n\rangle\}$ can be called the set of Fock states for pseudo-Hermitian boson system (shortly *pseudo Fock states*). In terms of the projectors on these states the pseudo-boson ladder operators b, $\tilde{b}$ can be represented as follows

$$\begin{array}{ll} b = \sum_n \sqrt{n}|\psi_{n-1}\rangle\,\langle\varphi_n|, & \tilde{b} = \sum_n \sqrt{n}|\psi_n\rangle\,\langle\varphi_{n-1}|, \\ b' = \sum_n \sqrt{n}|\varphi_{n-1}\rangle\,\langle\psi_n|, & b^\dagger = \sum_n \sqrt{n}|\varphi_n\rangle\,\langle\psi_{n-1}|. \end{array} \tag{12}$$

Now consider the operator [6]

$$\eta = \sum_n |\varphi_n\rangle\,\langle\varphi_n|. \tag{13}$$

This is Hermitian, positive and invertible operator, $\eta^{-1} = \sum_n |\psi_n\rangle\,\langle\psi_n|$. From the above expressions of η and η^{-1} one can see that $|\varphi_n\rangle = \eta|\psi_n\rangle$.

Finally one can easily check (using (12)) and (13) that $\tilde{b}$ is η-pseudo-adjoint to b, b' is η^{-1}-pseudo-adjoint to $b^\dagger$,

$$\tilde{b} = \eta^{-1}b^\dagger\eta, \quad b' = \eta b\eta^{-1}, \tag{14}$$

and N and N' are η- and η^{-1} pseudo-Hermitian: $N^\# := \eta^{-1}N^\dagger\eta = N$, $(N')^\# := (\eta')^{-1}(N')^\dagger\eta' = N'$, $\eta' = \eta^{-1}$.

4 Pseudo-boson coherent states

We define coherent states (CS) for the pseudo-Hermitian boson systems as eigenstates of the corresponding pseudo-boson annihilation operators. In this aim we introduce the pseudo-unitary displacement operator $D(\alpha) = \exp(\alpha b^{\#} - \alpha^* b)$ and construct eigenstates of b as displaced ground state $|0\rangle$,

$$b|\alpha\rangle = \alpha|\alpha\rangle, \quad \rightarrow \quad |\alpha\rangle = D(\alpha)|0\rangle, \tag{15}$$

where $\alpha \in C$. Using BCS formula one gets the expansion

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |\psi_n\rangle. \tag{16}$$

The structures of the above two formulas are the same as those for the case Hermitian boson CS (the Glauber canonical CS [16]), but the properties of our $D(\alpha)$ and $|\alpha\rangle$ are different. First note that $D(\alpha)$ is not unitary. Therefore $|\alpha\rangle$ are not normalized. Second, the set $\{|\alpha\rangle; \alpha \in C\}$ is not overcomplete (since $|\psi_n\rangle$ are not orthogonal).

The way out of this impasse is to consider the eigenstates of the dual ladder operator $b' = (b^{\#})^\dagger$, which take the analogous to (15) form,

$$b'|\alpha\rangle' = \alpha|\alpha\rangle', \quad \rightarrow \quad |\alpha\rangle' = D'(\alpha)|0\rangle', \tag{17}$$

where $D'(\alpha) = \exp(\alpha b^\dagger - \alpha^* b')$ is the complementary pseudo-unitary displacement operator. Therefore the eigenstates $|\alpha\rangle'$ are again non-normalized and do not form overcomplete set. However they are *bi-normalized* to $|\alpha\rangle$ (that is $\langle\alpha|\alpha\rangle' = 1$) and the system $\{|\alpha\rangle, |\alpha\rangle'; \alpha \in C\}$ is *bi-overcomplete* in the following sense

$$\frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle' \, \langle\alpha| = 1, \quad \frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle \, {}'\langle\alpha| = 1. \tag{18}$$

It is this bi-overcomplete set that we call *pseudo-boson CS*, or CS of pseudo-boson systems. More precisely they are η- pseudo-boson CS. When $\eta = 1$ these states recover the famous Glauber CS $|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle$, where a, $a^\dagger$ are canonical boson annihilation and creation operators.

5 Example

In this section we illustrate the above described scheme of construction of pseudo-boson Fock states and CS on the example of the following one-parameter family of non-Hermitian operators,

$$\begin{aligned} b(s) &= a + s a^\dagger, \\ \tilde{b}(s) &= s a + (1 + s^2) a^\dagger, \end{aligned} \tag{19}$$

where $s \in (-1, 1)$ and a, $a^\dagger$ are Bose annihilation and creation operators: $[a, a^\dagger] = 1$. It is clear that $b(0) = a$, $\tilde{b}(0) = a^\dagger$ and $\tilde{b}(s) \neq b^\dagger(s)$. In this way the parameter s could be viewed as a measure of deviation of $b(s)$ and $b^{\#}(s)$ from the canonical boson operators a and $a^\dagger$.

5.1 Pseudo-boson Fock state wave functions

The $b(s)$- and b'-vacuums $|0\rangle$ and $|0\rangle'$ do exist. Using the coordinate representation of $b(s)$ and $\tilde{b}(s)$,

$$b(s) = \frac{1}{\sqrt{2}}\left((1+s)x - (1-s)\frac{d}{dx}\right), \tag{20}$$

$$\tilde{b}(s) = \frac{1}{\sqrt{2}}\left((s+1+s^2)x - (s-1-s^2)\frac{d}{dx}\right), \tag{21}$$

we find the wave functions of $|0\rangle$ and $|0\rangle'$ ($\mathcal{N}(s) = \left(\pi(1-s)(1-s+s^2)\right)^{-\frac{1}{4}}$):

$$\psi_0(x,s) = \mathcal{N}(s)\exp\left(-\frac{1+s}{2(1-s)}x^2\right), \tag{22}$$

$$\varphi_0(x,s) = \mathcal{N}(s)\exp\left(-\frac{1+s+s^2}{2(1+s^2-s)}x^2\right), \tag{23}$$

The above two wave functions are bi-normalized, that is $\langle\varphi_0|\psi_0\rangle = 1$, if the parameter s is restricted in the interval $(-1,1)$, that is $-1 < s < 1$.

Therefore, according to Proposition 1 (and the related development in the previous section) $\tilde{b}(s)$ is η-pseudo-adjoint to $b(s)$ and (for $s^2 < 1$) the bi-orthonormalized Fock states and bi-overcomplete CS can be explicitly realized. The wave functions of the pseudo-boson Fock states (6) and (10) are obtained in the following form:

$$\begin{aligned}\psi_n(x,s) &= \frac{1}{\sqrt{2^n n!}}P_n(x,s)\psi_0(x,s),\\ \varphi_n(x,s) &= \frac{1}{\sqrt{2^n n!}}Q_n(x,s)\varphi_0(x,s),\end{aligned} \tag{24}$$

where $P_n(x,s)$, $Q_n(x,s)$ are polynomials of degree n in x, defined by means of the following recurrence relations

$$\begin{aligned}P_n &= \frac{2}{1-s}xP_{n-1} + (n-1)\frac{2(s-s^2-1)}{1-s}P_{n-2},\\ Q_n &= \frac{2}{1+s^2-s}xQ_{n-1} + (n-1)\frac{2(s-1)}{1+s^2-s}Q_{n-2}.\end{aligned} \tag{25}$$

For the first three values of n, $n = 0,1,2$, the polynomials $P_n(x,s)$ and $Q_n(x,s)$ read:

$$\begin{aligned}&P_0 = 1, \quad P_1 = \frac{2}{1-s}x, \qquad P_2 = \frac{4}{(1-s)^2}x^2 + \frac{2(s-s^2-1)}{1-s},\\ &Q_0 = 1, \quad Q_1 = \frac{2}{1-s+s^2}x, \;\; Q_2 = \frac{4}{(1-s+s^2)^2}x^2 + \frac{2(s-1)}{1-s+s^2}.\end{aligned} \tag{26}$$

At $s = 0$ these two polynomials $P_n(x,0)$ and $Q_n(x,0)$ coincide and recover the known Hermite polynomials $H_n(x)$. Therefore $P_n(x,s)$ and $Q_n(x,s)$ can

be viewed as two different generalizations of $H_n(x)$. They are not orthogonal. Instead of the orthogonality they satisfy the *bi-orthonality* relations, the weight function being $w(x,s) = \psi_0(x,s)\varphi_0(x,s)$,

$$\int P_n(x,s)Q_m(x,s)\,\psi_0(x)\varphi_0(x) = n!2^n\delta_{nm}. \tag{27}$$

Therefore $P_n(x,s)$ and $Q_n(x,s)$ realize *bi-orthogonal generalization* of $H_n(x)$. The relations between these three types of polynomials may be illustrated by the following diagram (where $\mu(s) = 1/(1-s)(1-s+s^2)$):

$$\begin{tikzcd}
& H_n(x) \arrow[dl] \arrow[d] \arrow[r] & \int H_n H_m e^{-x^2} dx = \sqrt{\pi}2^n n!\,\delta_{nm} \\
P_n(x) & Q_n(x) \arrow[r] & \int P_n Q_m e^{-\mu(s)x^2} dx = \mathcal{N}^{-2}(s)2^n n!\,\delta_{nm}
\end{tikzcd}$$

It is worth emphasizing that the above bi-orthogonal generalization of $H_n(x)$ is not unique. If instead of (19), we take another pair of operators satisfying the Proposition 1, say

$$\begin{aligned} b_2(s) &= a + sa^\dagger, \\ \tilde{b}_2(s) &= -sa + (1-s^2)a^\dagger, \end{aligned} \tag{28}$$

and apply the above described scheme, we would get another similar pair of bi-orthogonal polynomials.

5.2 Pseudo-boson CS wave functions

In coordinate representation equations (15), (17) for the eigenstates of $b(s)$ and $b'(s) = b^{\#^\dagger}(s)$ ($b(s)$ being given in (20)) lead to the following wave functions for any $\alpha \in C$,

$$\psi_\alpha(x,s) = \mathcal{N}(s,\alpha)\exp\left[-\frac{1+s}{2(1-s)}x^2 + \frac{\sqrt{2}\alpha}{1-s}x\right], \tag{29}$$

$$\varphi_\alpha(x,s) = \mathcal{N}'(s,\alpha)\exp\left[-\frac{1+s+s^2}{2(1-s+s^2)}x^2 + \frac{\sqrt{2}\alpha}{1-s+s^2}x\right], \tag{30}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are bi-normalization constants. Up to constant phase factors they are determined by the bi-normalization condition $\langle\psi_\alpha|\varphi_\alpha\rangle = 1$. We put $\alpha = \alpha_1 + i\alpha_2$ and find

$$\begin{aligned} (N^*N')^{-1} &= \int \exp\left[-\tfrac{x^2}{(1-s)(1-s+s^2)} + \sqrt{2}x\tfrac{(2-2s+s^2)\alpha_1 + i\alpha_2 s^2}{(1-s)(1-s+s^2)}\right]dx \\ &= \sqrt{\pi(1-s)(1-s+s^2)}\exp\left[\tfrac{\left(2\alpha_1 - 2\alpha_1 s + \alpha_1 s^2 + i\alpha_2 s^2\right)^2}{2(1-s)(1-s+s^2)}\right]. \end{aligned} \tag{31}$$

At $s=0$ we get, up to a constant phase factor, $\mathcal{N}=\mathcal{N}'=(1/\pi^{1/4})\exp(-\alpha_1^2)$. At $s=0$ both $\psi_\alpha(x,0)$ and $\varphi_\alpha(x,0)$ recover the wave functions of Glauber canonical CS.

The constructed Fock states and CS are time independent, and can be used as initial states (initial conditions) of pseudo-boson systems. Important question arises of the *temporal stability* of these states. In analogy with the case of Hermitian mechanics we define temporal stability of a given set of states by the requirement the time evolved states to belong to the same set. This means that, up to a time-dependent phase factor, the time-dependence of any state from the initial set should be included in the time-dependent parameters only. Clearly the time dependent parameters should remain in the same domain as defined initially.

For our CS the temporal stability means that the time evolved wave functions $\psi_\alpha(x,s,t)$, $\varphi_\alpha(x,s,t)$ should keep the form

$$\begin{aligned}\psi_\alpha(x,s,t) &= e^{i\chi(t)}\psi_{\alpha(t)}(x,s),\\ \varphi_\alpha(x,s,t) &= e^{i\chi'(t)}\varphi_{\alpha(t)}(x,s)),\end{aligned}\tag{32}$$

where $\chi(t),\chi'(t)\in R$, $\alpha(t)\in C$ and $s(t)^2<1$. It is clear, that if the time evolved CS obey (32) they remain eigenstates of the same ladder operators $b(s)$ and $b'(s)$. (Let us recall at this point that we are in the Schrödinger picture, where operators are time-independent). As an illustration consider now the time evolution of CS governed by the simple pseudo-Hermitian Hamiltonian

$$H_{\rm po}=\omega\left(b^{\#}(s)b(s)+\frac{1}{2}\right),\tag{33}$$

where $b^{\#}(s)=\eta^{-1}b^\dagger(s)\eta$, $\omega\in R_+$. System with Hamiltonian of the type (33) should be called *pseudo-Hermitian oscillator*. At $s=0$ it coincides with the Hermitian harmonic oscillator of frequency ω. In pseudo-Hermitian mechanics the time evolution of initial $\psi_\alpha(x,s)$ and $\varphi_\alpha(x,s)$, by definition, is given by

$$\begin{aligned}\psi_\alpha(x,s,t) &= e^{-iHt}\psi_\alpha(x,s),\\ \varphi_\alpha(x,s,t) &= e^{-iH^\dagger t}\varphi_\alpha(x,s),\end{aligned}\tag{34}$$

For $H=H_{\rm po}$ equations (34) produce

$$\begin{aligned}\psi_\alpha(x,s,t) &= e^{-i\omega t/2}\psi_{\alpha(t)}(x,s),\\ \varphi_\alpha(x,s,t) &= e^{-i\omega t/2}\varphi_{\alpha(t)}(x,s),\quad \alpha(t)=\alpha e^{-i\omega t},\end{aligned}\tag{35}$$

which shows that the evolution of CS, governed by the pseudo-Hermitian oscillator Hamiltonian $H_{\rm po}$ is temporally stable. Comparing (35) and (32) we see that for the system (33) the time-evolved CS remain stable with constant s, $\alpha(t)=e^{-i\omega t}$ and $\chi=-i\omega t/2$.

Finally it is worth noting that the pseudo-Hermitian Hamiltonian $H_{\rm po}$ has *unbroken* PT-symmetry [1]. Indeed, from (20) and (20) it follows that $PTH_{\rm po}PT=H_{\rm po}$, and from (24) and (25) it follows that all eigenstates of $H_{\rm po}$ (and of $H_{\rm po}^\dagger$ as well) are eigenvectors of PT with eigenvalue $+1$ or -1.

Conclusion

We have shown that if the commutator of two non-Hermitian operators b and $\tilde{b}$ equals 1 and b annihilates a state ψ_0 then $\tilde{b}$ is η-pseudo-Hermitian agjoint $b^\#$ of b and $b^\dagger$ is η^{-1}-pseudo-adjoint of $(\tilde{b}^\#)^\dagger$. Eigenstates of the pseudo-boson number operator $b^\# b$ and its adjoint $b^\dagger(b^\#)^\dagger$ form a bi-orthonormal system of pseudo-boson Fock states, while eigenstates of b and its complementary lowering operator $b' = (b^\#)^\dagger$ are shown to form *bi-normalized and bi-overcomplete* system.This system of states is regarded as system of coherent states (CS) for pseudo-Hermitian bosons. We have provided a simple one-parameter family of ladder operators $b(s)$ and $\tilde{b}(s)$ that possess the above described properties and constructed the wave functions of the related Fock states and CS. Fock state wave functions are obtained as product of an exponential of a quadratic form of x and one of the two new polynomials $P_n(x)$, $Q_n(x)$ that are bi-orthogonal and at $s = 0$ recover the standard Hermite orthogonal polynomials. The pseudo-boson CS are shown to be temporally stable for the pseudo-boson oscillator Hamiltonian $\omega\left(b^\#(s)b(s) + 1/2\right)$.

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